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# Generalized Entropy Composition with Different q Indices: A Trial

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K.Sasaki and M.Hotta
*Department of Physics, Tohoku University,*
*Sendai 980-8578, Japan*

**Abstract**

We analyze systematically composable composite entropy of two Tsallis subsystems with different q indices. H-theorem and thermal balance relation are commented. This report is based mainly on our recent paper [1].

# 1 Introduction

It has already been pointed out that the Tsallis statistical mechanics may be useful for explaining many anomalous systems. Nevertheless it should be said that its fundamental understanding has not been achieved yet, because a lot of crucial problems still remain open.

For example, why extended entropies take the Tsallis form ? What determines its q index ? These topics are certainly a part of the most important questions for the formulation.

On the other hand, we believe, it is also an essential problem whether the second law of the thermodynamics of composite systems with *different* q indices holds or not. If one of the subsystems takes $q = 1$, it behaves as an ordinary system and the sub-entropy is just Boltzmann-Gibbs. Then increase of the composite entropy may be worth guaranteeing that the Tsallis form is actually a physically relevant entropy. If one takes a small ordinary subsystem compared to the Tsallis subsystem and interaction between the systems is negligibly weak, we can regard the ordinary part as a thermometer. The setup will enable us to discuss what is an observable temperature of the Tsallis subsystem. Significance of the problem was first, as far as I know, pointed out exhaustively by Rajagopal[2] and a conjecture on the thermal balance for the different q-indices case is discussed by Tsallis[3].

There exist many other motivations to think composite systems with different q indices. For example[4], non-neutral electron gas is often argued as a Tsallis system with $q \sim 0.5$. The plasma electrons possess spin degree of freedom besides the spatial one. Thus if one wants to incorporate the spin thermal fluctuation in the external magnetic field, the composite entropy form with both $q_{spatial} \sim 0.5$ and $q_{spin}(\neq q_{spatial})$ included will be invoked. Statistical aspects of internal degree of freedom (iso-spin, baryon charge, lepton charge and so on) of self-gravitating systems may also require such a composite entropy.

In order to write down explicitly the composite entropy form with different q indices, guiding principles for the entropy should be requested just as symmetries play crucial roles when actions of general relativity and quantum field theories are fixed. However for the nonextensive statistical mechanics any principles are not yet established by physical evidence. Thus we must

investigate all possibilities that could appear in Nature, however, this is too high to access.

Therefore, as a strategy, we firstly divide all possible composite forms with two categories, composable and noncomposable and, as a first step, concentrate on the composable entropy category. The second category, non-composable entropy, looks technically hard to analyze exhaustively, so we keep the non-composable investigation beyond our scope of this report.

Tsallis is the first person who was aware that the composable property is not automatically equipped and emphasized its significance in the context of the same-q case, as follows[5].

*It concerns the non-trivial fact that the entropy $S(A+B)$ of a system composed of two independent subsystems $A$ and $B$ can be calculated from the sub-entropies $S(A)$ and $S(B)$, without any need of microscopic knowledge about $A$ and $B$, other than the knowledge of some generic universality class, herein the nonextensive universality class, represented by the entropic index $q$, i.e., without any knowledge about the microscopic possibilities of $A$ and $B$ nor their associated probabilities.*

Also Joichi and one of the authors[6] demonstrate explicitly powerful ability of composability in determination of generalized entropic forms. For example, uniqueness of the Tsallis entropy has been shown by imposing composability on a rather generic entropy form as

$$S = C + \sum_i \phi(p_i). \tag{1}$$

In this paper, we analyze in detail composable composite entropy of two Tsallis subsystems with different q values. We also show its H-theorem nature and thermal balance relation (the zero-th law) in the sense proposed by Abe[7]. This report is based on a recent work [1]. To follow the analysis in more detail, see the original paper.

# 2 Composable Entropy

For the case with the same sub-indices $q$, the Tsallis canonical distribution

of the composite system is obtained by maximizing the following action [8].

$$\begin{aligned}\tilde{S} &= S_{A+B} - \alpha \left( \sum_{i=1}^{N} \sum_{j=1}^{M} P_{ij} - 1 \right) \\ &\quad - \beta \left( \sum_{i=1}^{N} \sum_{j=1}^{M} E_{ij} P_{ij} - < E > \right), \end{aligned} \tag{2}$$

where $\alpha$ and $\beta$ are Lagrange multipliers and generate the unitary condition:

$$\sum_{i=1}^{N} \sum_{j=1}^{M} P_{ij} = 1, \tag{3}$$

and the total energy constraint:

$$\sum_{i=1}^{N} \sum_{j=1}^{M} E_{ij} P_{ij} = \sum_{i=1}^{N} \sum_{j=1}^{M} (E_i^A + E_j^B) P_{ij} =< E > . \tag{4}$$

Here $P_{ij}$ denotes the escort function of the probability $p_{ij}$ as

$$P_{ij} = \frac{(p_{ij})^q}{\sum_{k=1}^{N} \sum_{l=1}^{M} (p_{kl})^q}, \tag{5}$$

and the composite entropy $S_{A+B}$ is just given as a standard Tsallis form:

$$\begin{aligned} S_{A+B} &= -\frac{1}{1-q} \left[ 1 - \sum_{i=1}^{N} \sum_{j=1}^{M} (p_{ij})^q \right] \\ &= -\frac{1}{1-q} \left[ 1 - \left( \sum_{i=1}^{N} \sum_{j=1}^{M} (P_{ij})^{\frac{1}{q}} \right)^{-q} \right]. \end{aligned} \tag{6}$$

We extend the above formulation in order to include the cases with different q sub-indices. Treating the escort function $P_{ij}$ as the fundamental variable for variational procedures admits keeping the Lagrange-multiplier terms in eqn (2) unchanged, because they do not have explicit q dependence. Thus we modify the composite entropy form so as to depend on two positive sub-indices $q_A$ and $q_B$.

Here let us impose composability on the composite entropy with two indices:

$$S_{A+B}(P_{ij} = P_i^A P_j^B) = \lambda(S_A, S_B), \tag{7}$$

where $\lambda$ is arbitrary function of $S_A$ and $S_B$ and

$$S_A = -\frac{1}{1-q_A}\left[1 - \left(\sum_{i=1}^{N}(P_i^A)^{\frac{1}{q_A}}\right)^{-q_A}\right], \tag{8}$$

$$S_B = -\frac{1}{1-q_B}\left[1 - \left(\sum_{j=1}^{M}(P_j^B)^{\frac{1}{q_B}}\right)^{-q_B}\right]. \tag{9}$$

Then it can be proven that the most general form satisfying composability (7) is given as

$$S_{A+B} = \Omega(X_a, \bar{X}_b; r_A, r_B), \tag{10}$$

where

$$X_1 = \sum_{j=1}^{M}\left(\sum_{i=1}^{N}(P_{ij})^{r_A}\right)^{\frac{r_B}{r_A}}, \tag{11}$$

$$X_2 = \sum_{j=1}^{M}\left(\sum_{i=1}^{N}P_{ij}\right)^{r_B}, \tag{12}$$

$$X_3 = \sum_{j=1}^{M}\left(\sum_{i=1}^{N}(P_{ij})^{r_A}\right)^{\frac{1}{r_A}}, \tag{13}$$

$$\bar{X}_1 = \sum_{i=1}^{N}\left(\sum_{j=1}^{M}(P_{ij})^{r_B}\right)^{\frac{r_A}{r_B}}, \tag{14}$$

$$\bar{X}_2 = \sum_{i=1}^{N}\left(\sum_{j=1}^{M}P_{ij}\right)^{r_A}, \tag{15}$$

$$\bar{X}_3 = \sum_{i=1}^{N}\left(\sum_{j=1}^{M}(P_{ij})^{r_B}\right)^{\frac{1}{r_B}}, \tag{16}$$

and

$$r_A = \frac{1}{q_A}, \tag{17}$$

$$r_B = \frac{1}{q_B}. \tag{18}$$

Here $\Omega$ is arbitrary function with permutation symmetry between the subsystems $A$ and $B$:

$$\Omega(\bar{X}_b, X_a; r_B, r_A) = \Omega(X_a, \bar{X}_b; r_A, r_B). \tag{19}$$

Moreover we can introduce another composability when one constructs a grand composite system $(A+B)+(A+B)'$ of two composite systems $(A+B)$ and $(A+B)'$. Let us impose the following property on their entropies.

$$S_{(A+B)+(A+B)'} = \Lambda(S_{(A+B)}, S_{(A+B)'}), \tag{20}$$

where $\Lambda$ is an arbitrary function. This implies that the value of the grand entropy is fixed only by information of the composite entropies $S_{(A+B)}$ and $S_{(A+B)'}$. Then bi-composability is defined by realization of both the above two composabilities (7) and (20). Recall here that the Tsallis entropy actually satisfies the bi-composability when the $q$ indices of the sub-systems are the same. Here we should also stress that the concept of bi-composability is associated with a set of the two *simultaneous* equations (7) *and* (20), thus different notion from the original composability, which implies a single relation as eqn (7), or eqn (20). In fact some composable entropy forms which satisfy eqn (7) or eqn (20) do *not* show the bi-composability even when the subsystems are statistically independent. Also note that the there is no reason in general that the functional form $\Lambda$ coincides with the form of $\lambda$.

It is possible to write down the most generic form of the bi-composable entropy and the result is as follows.

$$S_{A+B} = F(\Delta; r_A, r_B), \tag{21}$$

where $F(x; r_A, r_B)$ is an arbitrary function and

$$\Delta = \prod_{a=1}^{3} (X_a)^{-\nu_a} \prod_{b=1}^{3} (\bar{X}_b)^{-\bar{\nu}_b} > 0. \tag{22}$$

Here the exponents $\nu_a$ and $\bar{\nu}_b$ are arbitrary constants in this level and supposed to be fixed by dynamical property of the system.

We can propose a simple and attractive toy model for the bi-composable entropy. If one assumes the Tsallis-type nonextensivity:

$$S_{(A+B)+(A+B)'} = S_{(A+B)} + S_{(A+B)'} + (1-Q)S_{(A+B)}S_{(A+B)'} \tag{23}$$

for the grand composite system, it is shown that the entropy must take the following form.

$$S_{A+B} = -\frac{1-\Delta}{1-Q}. \tag{24}$$

Here $Q$ behaves as a grand index of the composite system and is expected to be determined by some dynamical information of the system, just like usual q index. Later we call the simple model (24) Tsallis-type bi-composable entropy.

# 3 H-Theorem

Here we comment on H-theorem for the composable entropy. Unfortunately H-theorem does not hold in a strong sense for all the composable entropies in eqn (10). It is proven analytically that there exists a master equational dynamics in which some probability configurations give negative values of time-derivative of the composite entropies. However this fact may not be so significant for real physical systems, because the fixed-two-subindices picture does not always need to work when the total system is composed of two Tsallis systems with *different* q indices. Each original subindex is widely believed to be chosen dynamically for each isolated system. Thus it may happen in general that the interaction between the two subsystems drives the q values changed, or the q-deformed statistical picture itself gets broken and should be replaced by more microscopical pictures. Therefore it sounds plausible that the physical situations are somehow limited in which the total system can be regarded as a composite system of two independent q-deformed subsystems. Meanwhile it seems natural, at least, to consider that

the picture should work well when the interaction between the subsystems are negligibly weak, or the two subsystems are in near-equilibrium states. Actually H-theorem for such physically relevant cases can be exactly proven for a part of the bi-composable entropies. The Tsallis-type bi-composable entropies (24) which satisfy

$$r_B(r_B - r_A)\nu_1 + r_B(r_B - 1)\nu_2 - (r_A - 1)\nu_3 = 0, \tag{25}$$

$$r_A(r_A - r_B)\bar{\nu}_1 + r_A(r_A - 1)\bar{\nu}_2 - (r_B - 1)\bar{\nu}_3 = 0 \tag{26}$$

and

$$\frac{r_B(r_A - 1)\nu_1 + r_A(r_B - 1)\bar{\nu}_1 + (r_A - 1)\nu_3 + (r_B - 1)\bar{\nu}_3}{1 - Q} > 0 \tag{27}$$

do not decrease in time for the near-microcanonical-equilibrium case:

$$P_{ij}(t) = \frac{1}{NM} + \epsilon_{ij}(t), \tag{28}$$

where $\epsilon_{ij}$ is infinitesimal deviation from the equipartition distribution. It is also noticed that for the negligibly-weak interaction case:

$$P_{ij}(t) = P_i^A(t) P_j^B(t), \tag{29}$$

the Tsallis-type bi-composable entropy satisfying (25),(26) and (27) preserves H-theorem. It is proven that if

$$c_A = \frac{1 - q_A}{1 - Q}\left(r_B\nu_1 + r_A\bar{\nu}_1 + r_A\bar{\nu}_2 + \nu_3\right) > 0, \tag{30}$$

$$c_B = \frac{1 - q_B}{1 - Q}\left(r_B\nu_1 + r_A\bar{\nu}_1 + r_B\nu_2 + \bar{\nu}_3\right) > 0 \tag{31}$$

hold, the entropy has non-negative time-derivative for the evolution (29). Using the relations (25) and (26), the conditions (30) and (31) can be rewritten into the same condition and that is just the third relation (27).

It is worth noting that for a rather general bi-composable entropy defined by use of an arbitrary function $G(x)$ monotonically increasing as

$$S_{A+B} = G\left(-\frac{1 - \Delta}{1 - Q}\right), \tag{32}$$

the H-theorem still holds in the cases (28) and (29) if the relations (25), (26) and (27) are simultaneously satisfied. For the entropy (32), we discuss next Abe's thermal balance relation [7], that is, the zero-th law of the thermodynamics.

## 4 Thermal Balance Relation

It turns out that the nonextensivity of the bi-composable entropy (32) is given as follows.

$$
\begin{aligned}
S_{A+B} &= G\left(-\frac{1-\bar{\Delta}}{1-Q}\right), \\
\bar{\Delta} &= [1+(1-q_A)S_A]^{r_B\nu_1+\nu_3+r_A\bar{\nu}_1+r_A\bar{\nu}_2} \\
&\quad \times [1+(1-q_B)S_B]^{r_A\bar{\nu}_1+\bar{\nu}_3+r_B\nu_1+r_B\nu_2} .
\end{aligned} \tag{33}
$$

Using the H-theorem conditions (25) and (26), the variation of the entropy can be written as

$$
\begin{aligned}
\delta S_{A+B} &= \frac{r_B(r_B-1)(\nu_1+\nu_2)+r_A(r_A-1)(\bar{\nu}_1+\bar{\nu}_2)}{1-Q}\Delta G' \\
&\quad \times \left[\frac{q_A}{1+(1-q_A)S_A}\delta S_A + \frac{q_B}{1+(1-q_B)S_B}\delta S_B\right].
\end{aligned} \tag{34}
$$

Along Abe's argument for the same q case[7], we take $\delta S_{A+B} = 0$ under the total energy conservation relation:

$$
\delta E_A + \delta E_B = 0 \tag{35}
$$

to get the thermal balance relation. The procedure is expected valid for thermodynamic-limit situations as in the same q case. The result is given as follows, independent from the functional form $G(x)$ and the value of $Q$.

$$
\frac{q_A}{1+(1-q_A)S_A}\frac{\delta S_A}{\delta E_A} = \frac{q_B}{1+(1-q_B)S_B}\frac{\delta S_B}{\delta E_B}. \tag{36}
$$

Note that this relation includes the Abe's balance relation [7] as a special case. Actually when $q_A = q_B = q$ is taken

$$
\frac{1}{1+(1-q)S_A}\frac{\delta S_A}{\delta E_A} = \frac{1}{1+(1-q)S_B}\frac{\delta S_B}{\delta E_B} \tag{37}
$$

is exactly reproduced. Also eqn (36) is consistent with a guessed relation [3] by Tsallis for the different q case.

If the system $A$ is taken as an ordinary Boltzmann-Gibbs system ($q_A = 1$), the sub-entropy is reduced into the BG form $S_{BG:A}$. Then the relation (36) with $q_A = 1$ is expressed as

$$\frac{\delta S_{BG:A}}{\delta E_A} = \frac{q_B}{1+(1-q_B)S_B}\frac{\delta S_B}{\delta E_B}. \tag{38}$$

Here it is a trivial fact that physical temperature $T_{phys}$ can be introduced for the system $A$ as follows.

$$\frac{1}{T_{phys}} = \frac{\delta S_{BG:A}}{\delta E_A}. \tag{39}$$

Therefore observable temperature $T_B$ of the Tsallis system $B$ should be defined as

$$\frac{1}{T_B} = \frac{q_B}{1+(1-q_B)S_B}\frac{\delta S_B}{\delta E_B}, \tag{40}$$

so as to preserve the zero-th law of thermodynamics:

$$\frac{1}{T_{phys}} = \frac{1}{T_B}. \tag{41}$$

Here we should stress that before our analysis no one argues explicitly presence of the numerator $q_B$ of the prefactor in the right-hand-side term of eqn (40). Due to the definition (40), the original relation (36), in which $q_A$ is not needed to take unit, can be interpreted as a generalized thermal balance as follows.

$$\frac{1}{T_A} = \frac{1}{T_B}. \tag{42}$$

The transitivity relation (42) looks quite plausible and attractive, though the derivation remains still heuristic.

# 5 Final Remarks

We discussed Tsallis entropy composition with different q indices in detail and showed several rigid conclusions for the composable entropy class. However we must agree that there still remain open problems. For example, what determines the functional form $\Omega$, or $F$, and the exponents $\nu_a$, $\bar{\nu}_b$ for the bi-composable entropy and the grand index $Q$ for the Tsallis-type model ? How calculate them ? These questions are perhaps as profound and difficult as the original q index problem of the Tsallis entropy. We just expect that they are determined by dynamical and somewhat microscopical information of physical systems.

However it is possible, as seen below, just to construct a simple and regular model as a special solution of the H-theorem problem.

$$\begin{aligned} S_{A+B} &= -\frac{r_A(r_A-1)^2+r_B(r_B-1)^2}{(r_A-1)(r_B-1)(r_A+r_B-2)} \\ &\times \left[1-\left(X_1^{r_B-1}\bar{X}_1^{r_A-1}\left[\frac{X_2}{\bar{X}_2}\right]^{r_A-r_B}\right)^{\frac{2-r_A-r_B}{2[r_A(r_A-1)^2+r_B(r_B-1)^2]}}\right]. \end{aligned} \tag{43}$$

This example succeeds in simplification of the form because it does not depend on $X_3$ and $\bar{X}_3$, while the H-theorem still holds. Also it has regular limits for both $r_A \to 1$ and $r_B \to 1$ independently, and no singularities for positive $q_A$ and $q_B$ region. (The limit $r_A + r_B \to 2$ is also regular.) Moreover it is easily confirmed that if one takes $q_A = q_B$, the form is reduced into the original Tsallis form. This toy model may be useful for future works to get more deeper intuition about physics of the generalized entropy composition.

Finally we want to comment that our analysis has not included at all the class of noncomposable composite entropy and it may be an interesting open problem to analyze the case.

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